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A PRESSURE-SMOOTHING SCHEME FOR INCOMPRESSIBLE FLOW PROBLEMS

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SUMMARY

A pressure-smoothing scheme for Stokes and Navier–Stokes flows of Newtonian fluids and for Stokes flow of Maxwell fluids is described. The stress deviator obtained from the calculated velocity field is substituted into the governing equilibrium equation. The resulting equation is then solved to obtain a new, smoothed pressure by a least square finite element method.

KEY WORDS Pressure-smoothing scheme Least square Finite element method Incompressible flow Newtonian and Maxwell fluids

INTRODUCTION

For finite element analyses of flow problems, it is often desirable to obtain accurate pressure fields. As an example, in polymer processing it is necessary to have particularly accurate pressures in order to predict conditions of the final product and also in order to design the processing equipment.

The pressure is determined by the continuity condition for incompressible flow and is often approximated with discontinuous functions. The penalty formulation may also be used, but this results in the elimination of the pressure from the finite element equations. Several smoothing schemes are available to obtain the pressure at the nodes used for velocity specification.[1–7] These include the method of expressing the equilibrium and continuity equations with Poisson's equation and solving for the pressure.

The method described in this study uses the stress deviator for smoothing the pressure. If we substitute the stress deviator for the stress in the equilibrium equation, the resulting equation has only the pressure as a variable. To this equation in pressure, a least square finite element method is applied. The linear set of finite element equations obtained is then easily solved for a smooth pressure.

GOVERNING EQUATIONS

We first present the governing equations for steady state incompressible Stokes and Navier–Stokes flows of Newtonian fluids and for Stokes flow of Maxwell fluids, in rectangular Cartesian co-ordinates.

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The equation of motion in terms of stress is

$$\sigma_{ij,j} + \rho X_i = \rho u_j u_{i,j}. \tag{1}$$

When creeping flows are under consideration, the right-hand side of equation (1) becomes zero.

The continuity equation (incompressible flow) is

$$u_{i,i} = \varepsilon_{ii} = 0 \tag{2}$$

and the constitutive relationships are

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma'_{ij} \quad \text{(Newtonian fluid)}, \tag{3}$$

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma'_{ij} + \frac{1}{2G}\overset{\nabla}{\sigma}{}'_{ij} \quad \text{(Maxwell fluid)}, \tag{4}$$

where

$$\sigma'_{ij} = \sigma_{ij} + p\delta_{ij}, \tag{5}$$

$$\overset{\nabla}{\sigma}{}'_{ij} = u_k \sigma'_{ij,k} - \sigma'_{ik}\omega_{kj} - \sigma'_{jk}\omega_{ki}, \tag{6}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{7}$$

$$\omega_{ij} = \tfrac{1}{2}(u_{j,i} - u_{i,j}), \tag{8}$$

$$p = -\tfrac{1}{3}\sigma_{ii}. \tag{9}$$

The boundary conditions are:

$$u_i = \bar{u}_i \quad \text{on } S_u, \tag{10}$$

$$v_j \sigma_{ij} = \bar{T}_i \quad \text{on } S_t, \tag{11}$$

$$\sigma_{ij} = \bar{\sigma}_{ij} \quad \text{on } S_\sigma \quad \text{(Maxwell fluid only)}. \tag{12}$$

Here σ_{ij} is the stress, ρ is the density, X_i is the body force per unit volume, u_i is the velocity, ε_{ij} is the strain rate, μ is the viscosity, G is the shear modulus of elasticity, p is the pressure, $\bar{u}_i$ is the specified velocity on S_u, $\bar{T}_i$ is the specified traction on S_t with unit outward normal vector v_j, and $\bar{\sigma}_{ij}$ is the specified stress on S_σ.

SOLUTION ALGORITHM FOR PRESSURE

The algorithm for the pressure-smoothing scheme is shown in Figure 1. We first solve the equilibrium and continuity equations for velocity. In this step the penalty function formulation is used for the Navier–Stokes problem with a Newton–Raphson scheme. For Stokes flow of Newtonian and Maxwell fluids we use the velocity–pressure formulation.

After the velocity and pressure fields have been obtained, we next calculate the stress deviator using the constitutive equation (3) or (4). The standard Galerkin method is applied for Newtonian fluids. For Maxwell fluids a least square type of finite element method is used in order to eliminate oscillations in the stress deviator field.[8] Because the constitutive equation of the Maxwell fluid is non-linear with respect to the stress deviator, iterations between equations (1), (2) and equation (4) are performed until convergence is achieved. The details of the algorithm used for this analysis can be found in References 8–11.

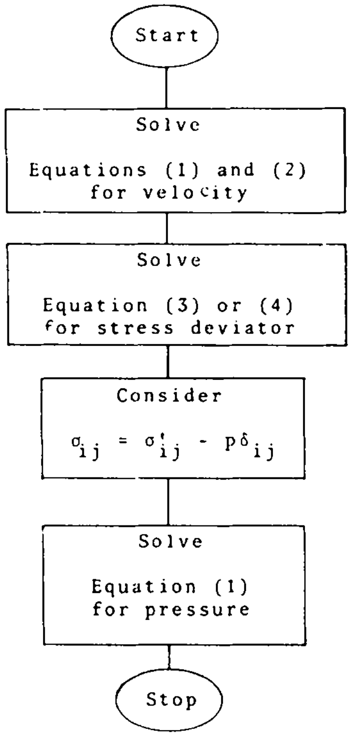

Figure 1. Solution algorithm for pressure

In order to obtain an equation for the pressure, we substitute the stress deviator into equation (1). Because the stress deviator is known, the stress can be expressed in terms of the pressure using equation (5). The least square finite element method is applied to the pressure equation obtained.

FINITE ELEMENT METHOD

We now outline the finite element method applied to flows of Newtonian and Maxwell fluids. Further details are shown in References 5 and 8.

The finite element approximations for velocity, pressure and stress deviator fields are taken as

$$u_i \approx N_\alpha u_{\alpha i}, \tag{13}$$

$$p \approx M_\alpha p_\alpha, \tag{14}$$

$$\sigma'_{ij} \approx L_\alpha \sigma'_{\alpha ij}. \tag{15}$$

Finite element method for velocity

Stokes flow of Newtonian and Maxwell fluids. Galerkin's method applied to equations (1) and (2) gives us

$$\int_v N_{\alpha,j} 2\mu\varepsilon_{ij} \mathrm{d}v - \int_v N_{\alpha,j} \frac{\mu}{G}(u_k \sigma'_{ij,k} - \sigma'_{ik}\omega_{kj} - \sigma'_{jk}\omega_{ki}) \mathrm{d}v - \int_v N_{\alpha,j}\delta_{ij} p \,\mathrm{d}v - \int_{s_t} N_\alpha T_i \mathrm{d}s - \int_v N_\alpha \rho X_i \mathrm{d}v = 0 \tag{16}$$

and

$$\int_v M_\alpha \varepsilon_{ii} \, \mathrm{d}v = 0. \tag{17}$$

If now the finite element approximations of equations (13), (14) and (15) are substituted in equations (16) and (17), the following simultaneous equations are obtained:

$$[K_1]\{U\} + [K_2]\{U\} - [G]\{p\} = \{F_1\}, \tag{18}$$

$$[G]^{\mathrm{T}}\{U\} = 0, \tag{19}$$

where K_1 represents the coefficients obtained from the first integral in equation (16), K_2 from the second integral, G from the third integral as well as in equation (17), and F_1 represents the coefficients obtained from the last two integrals in equation (16). The K_2 coefficients represent the non-linear terms introduced by the inclusion of the elastic strain rates and are non-symmetric. The U coefficients are the vector components of the nodal velocities and the p coefficients are the pressures.

For Newtonian fluids the second integral of equation (16) can be eliminated; in other words, the K_2 matrix does not appear in equation (18).

Six-node triangular isoparametric elements are used for approximating velocity and three-node elements with discontinuous linear approximations are used for pressure.

Navier–Stokes flow. In order to analyse Navier–Stokes flows, the penalty function formulation is incorporated in place of the pressure. Galerkin's method applied to equations (1) and (2) with weighting functions the same as the approximating functions gives us

$$\int_v N_{\alpha,j} 2\mu\varepsilon_{ij} \, \mathrm{d}v - \int_v N_\alpha \rho u_j u_{i,j} \, \mathrm{d}v + \int_v N_{\alpha,j}\delta_{ij}\lambda\varepsilon_{kk} \, \mathrm{d}v - \int_{S_t} N_\alpha T_i \, \mathrm{d}s - \int_v N_\alpha \rho X_i \, \mathrm{d}v = 0, \tag{20}$$

where λ is the penalty number. If the finite element approximation of equation (13) is substituted in equation (20), we have

$$[K_3]\{u\} + [C_1(u)]\{u\} + \lambda[K_4]\{u\} = \{F_2\}, \tag{21}$$

where K_3, $C_1(u)$, K_4 and F_2 represent the coefficients from the first and second integrals in equation (20), from the third integral and the last two integrals respectively.

When the Newton–Raphson scheme is used for solving equation (21), an incremental form may be written as

$$[K_3]\{\Delta u_{m+1}\} + [C_2(u_m)]\{\Delta u_{m+1}\} = \{F_2\} - [[K_3]\{u_m\} + [C_1(u_m)]\{u_m\}], \tag{22}$$

$$u_{m+1} = u_m + \Delta u_{m+1}, \tag{23}$$

where $[C_2(u_m)]\{\Delta u_{m+1}\}$ represents the derivative of the second term in equation (21) which can be written as

$$[C_2(u_m)] = \sum_{e=1}^{n} \int_{v_e} N_\alpha \rho \{(u_m)_{i,j} N_\beta + \delta_{ij}(u_m)_k N_{\beta,k}\} \, \mathrm{d}v_e \tag{24}$$

(n: number of elements).

A four-node approximation is used for velocity. For this approximation, the 2×2 Gauss–Legendre integration rule has been employed except for the third integral in equation (20). A one-point reduced integration is performed for the third integral which involves the penalty number.

Finite element method for stress deviator

Maxwell fluid flow. The least square method applied to equation (4) gives us

$$\int_v \left\{\frac{\mu}{G}(u_k L_{\alpha,k}) + L_\alpha\right\}\left\{\sigma'_{ij} + \frac{\mu}{G}(u_k \sigma'_{ij,k} - \sigma'_{ik}\omega_{jk} - \sigma'_{jk}\omega_{ki}\right\} dv$$

$$-\int_v \left\{\frac{\mu}{G}(u_k L_{\alpha,k}) + L_\alpha\right\} 2\mu\varepsilon_{ij} dv = 0. \tag{25}$$

Substitution of equation (15) for the stress deviator in equation (25) gives us

$$[K_\sigma^m]\{\sigma'\} = \{E^m\}, \tag{26}$$

where K_σ^m represents the coefficients obtained from the first integral of equation (25) and E^m represents the coefficients from the second integral. The σ' terms are components of the nodal stress deviator. A three-node approximation for the stress deviator is used.

Newtonian fluid flow. Galerkin's method applied to equation (3) gives us

$$\int_v L_\alpha \sigma'_{ij} dv - \int_v L_\alpha 2\mu\varepsilon_{ij} dv = 0. \tag{27}$$

Substitution of equation (15) for the stress deviator in equation (27) gives us

$$[K_\sigma^{ns}]\{\sigma'\} = \{E^{ns}\}, \tag{28}$$

where K_σ^{ns} represents the coefficients obtained from the first integral of equation (27) and E^{ns} represents the coefficients from the second integral. We note that equation (27) can also be obtained when μ/G of equation (25) becomes zero. A three-node approximation is used for Stokes flow and a four-node approximation for Navier–Stokes flow.

Finite element method for pressure

In order to transform discontinuous pressures, which appear in the third term of equation (18), to the nodal points, we may use a Galerkin finite element method simply by solving

$$\int_v [M]^T[M] dv\{p^*\} = \int_v [M] p\, dv, \tag{29}$$

where $\{p^*\}$ will be the nodal point pressure values and the p on the right-hand side is the discontinuous pressure obtained.

For the penalty function formulation we first calculate element pressures for the equation

$$p = -\lambda\varepsilon_{ii}. \tag{30}$$

We then transform the values obtained to the nodal points by equation (29).

The pressure fields can be smoothed without considering the discontinuous pressures obtained from equations (18) and (19) or the penalty number.

We now describe the pressure-smoothing scheme using the stress deviator. When the stress found from equation (5) is substituted in equation (1), we have

$$\rho u_j u_{i,j} - \sigma'_{ij,j} + \delta_{ij} p_{,j} - \rho X_i = 0. \tag{31}$$

Substituting the finite element approximation, equation (14), into equation (31) and considering the velocity and the stress deviator as constants, the least square finite element method gives us

$$\sum_{e=1}^{n} \int_{v_e} \delta_{ik} M_{\beta,k}(\rho u_j u_{i,j} - \sigma'_{ij,j} + \delta_{ij} M_{\alpha,j} p_\alpha - \rho X_i) \mathrm{d}v_e = 0. \tag{32}$$

In matrix notation we have

$$[K_4]\{p\} = \{F_3\}, \tag{33}$$

where

$$[K_4] = \sum_{e=1}^{n} \int_{v_e} \delta_{ij} M_{\beta,i} M_{\alpha,j} \mathrm{d}v_e, \tag{34}$$

$$\{F_3\} = \sum_{e=1}^{n} \int_{v_e} \delta_{ik} M_{\beta,k}(-\rho u_j u_{i,j} + \sigma'_{ij,j} + \rho X_i) \mathrm{d}v_e. \tag{35}$$

EXAMPLE PROBLEMS

We now use the above formulation to solve Stokes and Navier–Stokes flows of Newtonian fluids and Stokes flow of a Maxwell fluid.

Stokes flow of Newtonian fluid

As an example of the Stokes flow problem, the flow through an L-shaped channel is analysed. Figure 2 shows the geometry and the finite element mesh, and also shows the velocity distribution obtained from equations (18) and (19). Figure 3 shows the calculated pressure distributions. The solid line represents the pressures obtained by the present method and the dotted line by equation (29). It is clear that the pressures are smoothed by equation (32). On the other hand, equation (29) gives poor results.

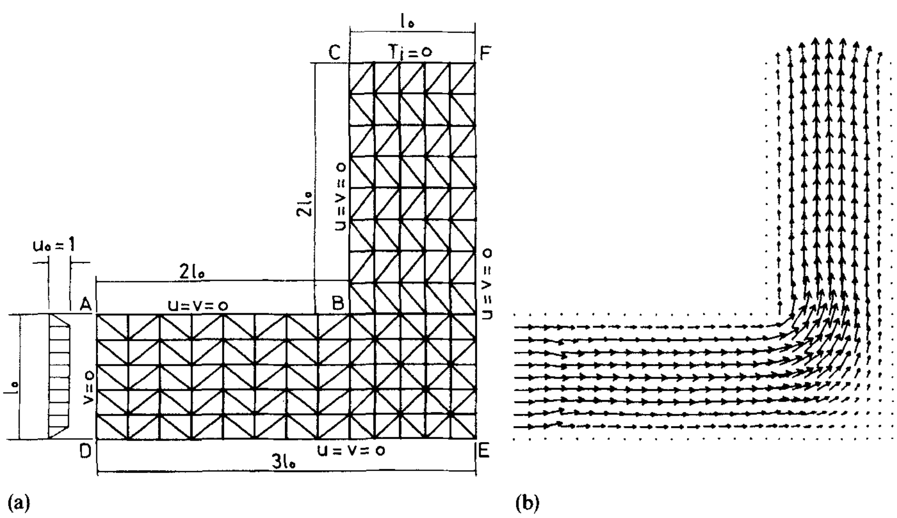

Figure 2. Stokes flow problem (L-shaped channel): (a) geometry and finite element mesh; (b) velocity distribution

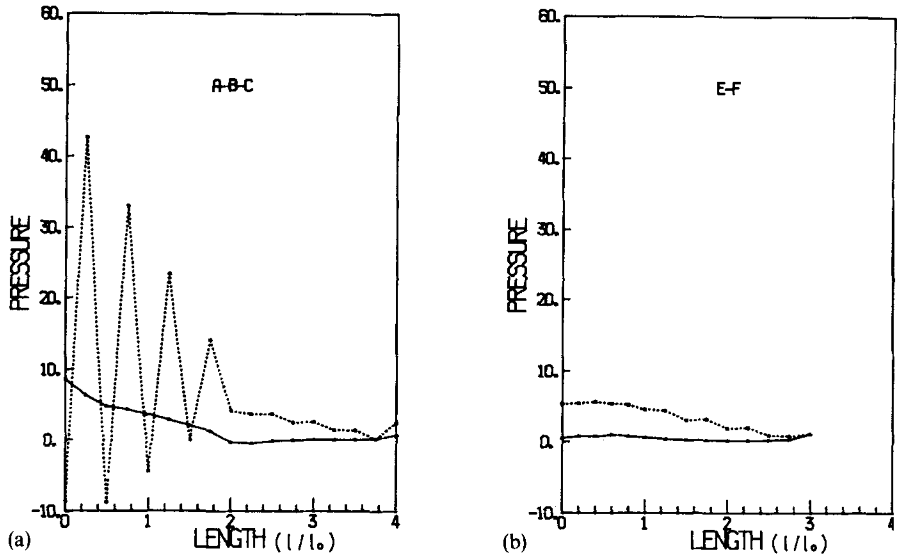

Figure 3. Pressure distribution of the L-shaped channel --------, equation (29); ———, present method; (a) pressure along A–B–C; (b) pressure along flow E–F

Navier–Stokes flow of Newtonian fluid

In order to test the algorithm, we first solve the Poiseuille flow problem shown in Figure 4. The smoothed pressure distribution with the present method is plotted in Figure 5. The penalty number $\lambda = 1.0 \times 10^6$ is chosen in this analysis.

For the next example, the driven cavity problem shown in Figure 6 is solved. The computed pressure contours with the present method for $R_e = 400$ are shown in Figure 7. The results agree well with those obtained by Yang and Atluri.[4]

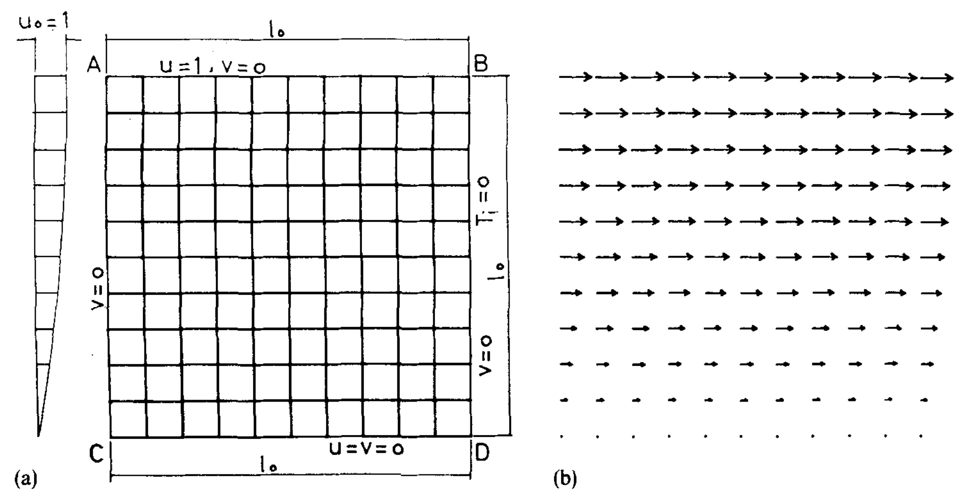

Figure 4. Navier–Stokes flow problem (Poiseuille flow); $Re = 1{\cdot}0$; (a) geometry and finite element mesh; (b) velocity distribution

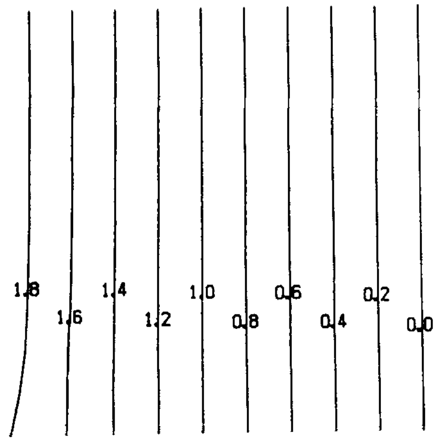

Figure 5. Pressure distribution of the Poiseuille flow; $\lambda = 1{\cdot}0 \times 10^6$

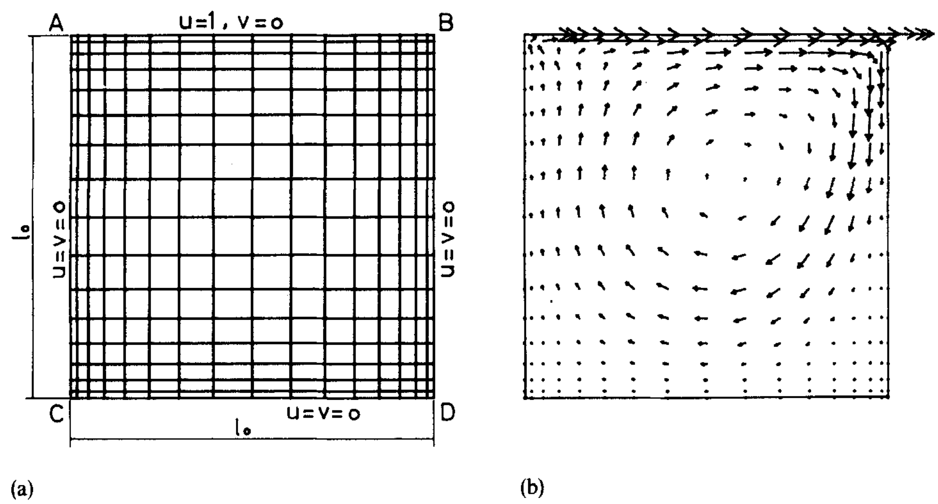

Figure 6. Navier–Stokes flow problem (cavity flow); $Re = 400$; (a) geometry and finite element mesh; (b) velocity distribution

Maxwell fluid flow

The last example analysed for the comparison of equation (29) and the present method is the rolling of an elasto-visco-plastic slab shown in Figure 8. The rollers are assumed to rotate at a constant angular velocity and to exhibit a no-slip interface with the slab. The upper symmetrical half of the slab is analyzed. Detailed discussion of this problem is given by Shimazaki and Shiojima.[8]

The pressure distributions along the centreline and the surface of the slab are plotted in Figures 9 and 10 respectively. From the figures, it is observed that the solutions obtained by equation (29) tend to exhibit larger pressure values than those obtained by the present method.

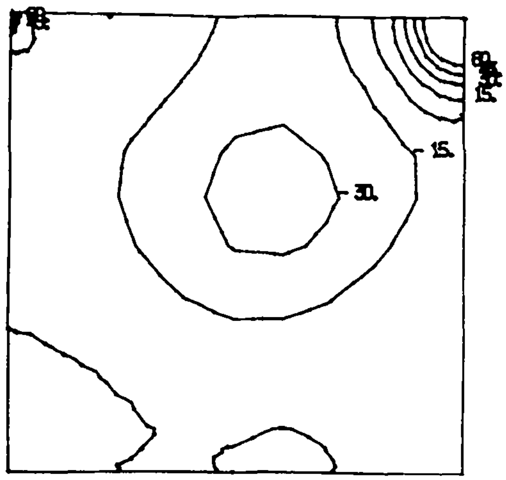

Figure 7. Pressure distribution of the cavity flow

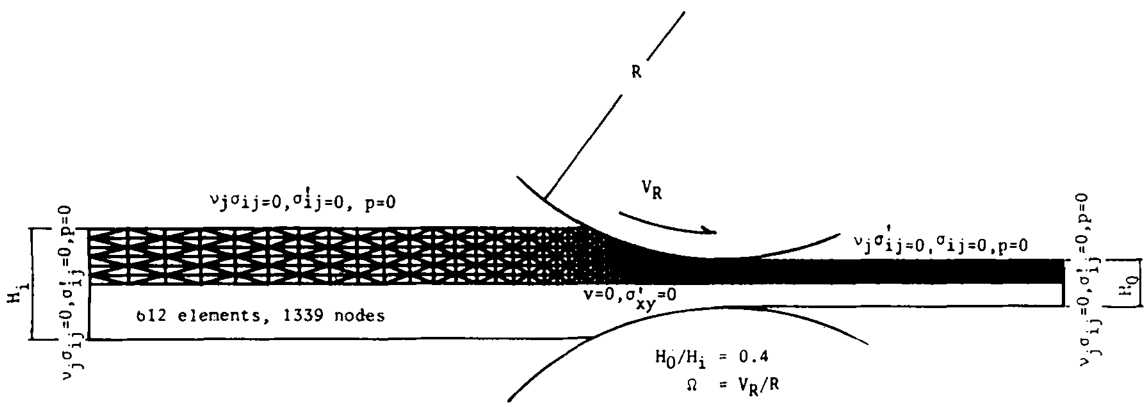

Figure 8. Maxwell fluid problem (rolling of an elasto-visco-plastic slab); geometry and finite element mesh

Along the surface, equation (29) leads to a spurious oscillation of the pressure in the vicinity of the roller exit. On the other hand, the present method has generally smoothed the pressures.

CONCLUSIONS

The pressure equation obtained from the equilibrium equation, in which the velocity and the stress deviator are considered constant, was solved by the finite element method as a simple and efficient pressure-smoothing scheme.

A comparison was made for Stokes and Navier–Stokes flows of Newtonian fluids and for Stokes flow of a Maxwell fluid between the present method and the method using the discontinuous pressure, which is calculated by the velocity–pressure formulation of the finite element method. It was shown that the present scheme generally smoothed the pressure field accurately.

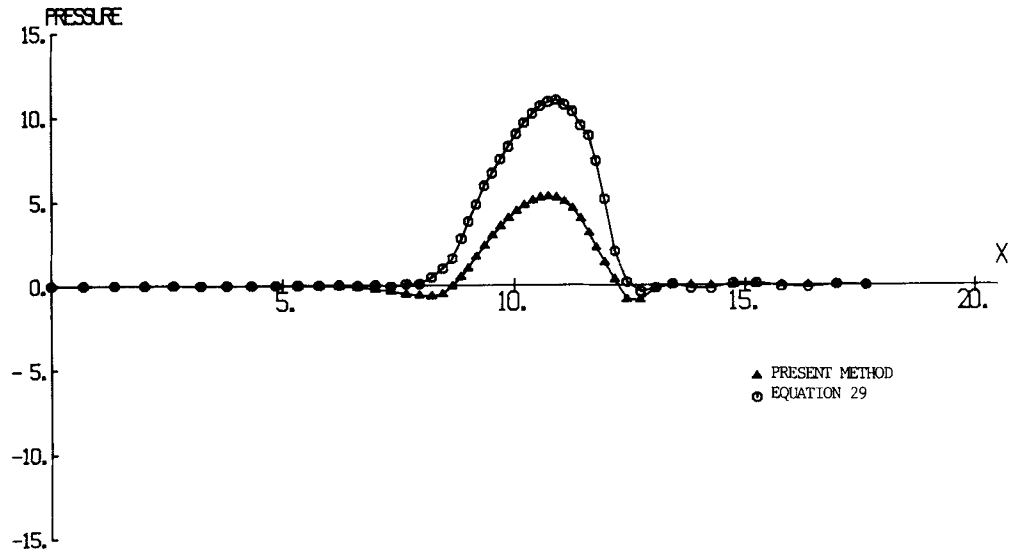

Figure 9. Pressure distribution along the centreline; unit = $\mu\Omega$ ($\mu\Omega/G = 0{\cdot}08$)

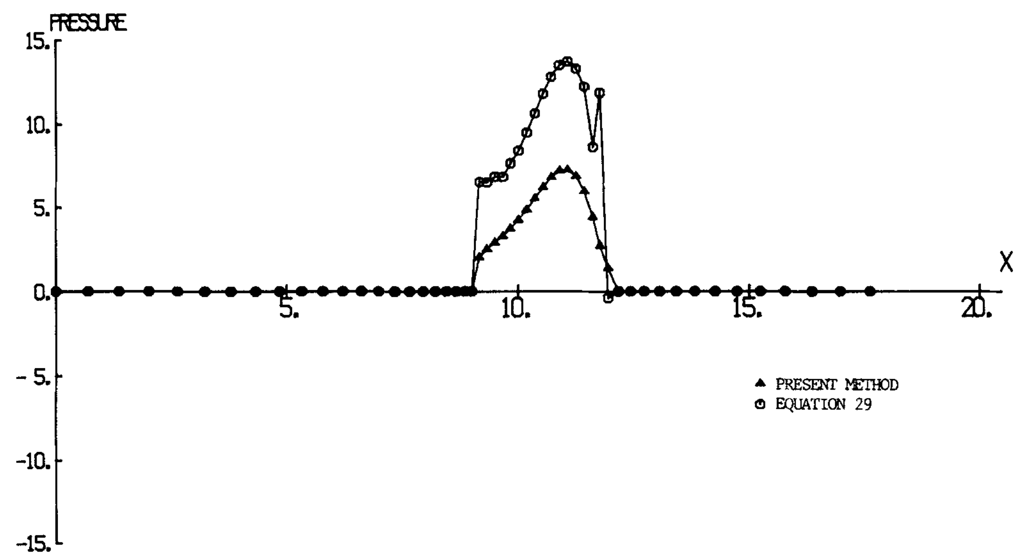

Figure 10. Pressure distribution along the surface; unit = $\mu\Omega$ ($\mu\Omega/G = 0{\cdot}08$)

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